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New properties of *(s,E,F)*-convex functions in the fourth sense

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Abstract – This study focuses on examining the class of *(s,E,F)*-convex functions in the fourth sense. Novel algebraic properties are introduced for this particular class of functions. Additionally, some relationships between these functions and other types of convex functions are explored. The results obtained in this work represent generalizations of previously established results.

Keywords – s-Convex Function, (E,F)-Convex Function, (s,E,F)-Convex Function, Generalized Convexity

I. INTRODUCTION

Convexity and its generalizations plays a key role in both pure and applied mathematics [1-7].

Definition 1. Let $V \subseteq \mathbb{R}$ be an interval. Then a function $\rho: V \longrightarrow \mathbb{R}$ is called convex if

$$\rho(tu+(1-t)v) \leq t\rho(u)+(1-t)\rho(v)$$

holds for all $u, v \in$ V and $t \in [0,1]$.

If the above inequality holds in the reverse direction, then ρ is said to be concave.

In [9], Youness defined the class of E-convex functions as follows:

Definition 2. A function $E: [u,v] \longrightarrow [u,v]$ where $[u,v] \subseteq \mathbb{R}$. A function $\rho: [u,v] \longrightarrow \mathbb{R}$ is called an E-convex function if

$$\rho(tE(\varkappa_1)+(1-t)E(\varkappa_2)) \leq t\rho(E(\varkappa_1))+(1-t)\rho(E(\varkappa_2))$$

holds for all $\varkappa_1, \varkappa_2 \in [u,v]$ and $t \in [0,1]$.

In [10], Jian introduced the notion of (E,F)-convex set and (E,F)-convex function as follows:

Definition 3. V is called (E,F)-convex set, if

$$tE(u)+(1-t)F(v) \in V$$

for all $u, v \in V$ and $t \in [0,1]$.

Definition 4. A function ρ is called (E,F)-convex function, if V is an (E,F)-convex set and

$$\rho(tE(u)+(1-t)F(v)) \leq t\rho(E(u))+(1-t)\rho(F(v))$$

holds for all $u, v \in$ V and $t \in [0,1]$.

In [2], Hudzik and Maligranda defined s-convex functions in the second sense as follows:

Definition 5. A function $\rho: [0,\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense if

$$\rho(tu+(1-t)v) \leq t^s\rho(u)+(1-t)^s\rho(v)$$

holds for all $u, v \in [0, \infty), t \in [0,1]$ and for some fixed $s \in (0,1]$.

In [11], the class of s-convex functions in the fourth sense is defined as follows:

Definition 6. A function $\rho: V \longrightarrow \mathbb{R}$ is said to be s-convex in the fourth sense if

$$\rho(tu + (1-t)v) \leq t^{\frac{1}{s}}\rho(u) + (1-t)^{\frac{1}{s}}\rho(v)$$

holds for all $u, v \in U$.

In [12], Saleh gave the definition of (s, E)-convex functions in the fourth sense as follows:

Definition 7. Let $E: \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $U \subseteq \mathbb{R}$ be a nonempty E-convex set. A function $\rho: U \longrightarrow \mathbb{R}$ is called an (s, E)-convex function in the fourth sense, if

$$\rho(tE(u) + (1-t)E(v)) \leq t^{\frac{1}{s}}\rho(E(u)) + (1-t)^{\frac{1}{s}}\rho(E(v))$$

holds for all $u, v \in U, s \in (0,1]$ and $t \in [0,1]$.

Recently, in [8] Özcan introduced the concept of (s, E, F)-convex function in the fourth sense and investigated some of its algebraic properties.

Definition 8. Let $E, F: \mathbb{R} \longrightarrow \mathbb{R}$ be two functions and $U \subseteq \mathbb{R}$ be a nonempty (E, F)-convex set. A function $\rho: U \longrightarrow \mathbb{R}$ is called (s, E, F)-convex function in the fourth sense if

$$\rho(tE(u) + (1-t)F(v)) \leq t^{\frac{1}{s}}\rho(E(u)) + (1-t)^{\frac{1}{s}}\rho\big(F(v)\big) \quad (1)$$

holds for all $u, v \in U, s \in (0,1]$ and $t \in [0,1]$.

If the above inequality holds in the reverse direction, then ρ is said to be (s, E, F)-concave function in the fourth sense.

Note that, the inequality (1) is equivalent to the inequality

$$\rho\big(t^s E(u) + (1-t)^s F(v)\big) \leq t\rho\big(E(u)\big) + (1-t)\rho\big(F(v)\big)$$

for all $t \in [0,1]$.

Remark 1. Definition 8 leads to the following results:

1. For $s = 1$, it coincides with the definition of (E, F)-convex function.

2. For $F = I$, it coincides with the definition of (s, E)-convex function in the fourth sense.

3. For $E = \mathrm{F} = I$, it coincides with the definition of s-convex function in the fourth sense.

4. For $s = 1$ and $E = \mathrm{F} = I$, it coincides with the definition of convex function.

II. Main Results

In this section we give some basic properties of (s, E, F)-convex function in the fourth sense.

Theorem 1. If $\rho_j: U \longrightarrow \mathbb{R}_$are (s, E, F)-convex functions in the fourth sense for $i = 1,2, \dots, m$, then $\rho: U \longrightarrow \mathbb{R}_$defined by $\rho = \max_{1 \leq j \leq m}\{\rho_j\}$ is an (s, E, F)-convex function in the fourth sense .

Proof. For all $u, v \in U$ and $t \in [0,1]$, one can write

$$\rho\big(tE(u) + (1-t)F(v)\big)$$
$$= \max_{1 \leq j \leq m}\{\rho_i\big(tE(u) + (1-t)F(v)\big)\}$$
$$\leq \max_{1 \leq j \leq m}\left\{\left(t^{\frac{1}{s}}\rho_j\big(E(u)\big) + (1-t)^{\frac{1}{s}}\rho_j\big(F(v)\big)\right)\right\}$$

$$\leq t^{\frac{1}{s}} \max_{1\leq j\leq m}\{\rho_j(E(u))\} + (1-t)^{\frac{1}{s}} \max_{1\leq j\leq m}\{\rho_j(F(v))\}$$
$$= t^{\frac{1}{s}}\rho(E(u)) + (1-t)^{\frac{1}{s}}\rho(F(v)).$$

Thus, $\rho = \max_{1\leq j\leq m}\{\rho_j\}$ is an (s,E,F)-convex function in the fourth sense .

Remark 2. For $F = I$, Theorem 1 coincides with Theorem 2.8 in [12].

Theorem 2. Let $0 < s_1 \leq s_2 \leq 1$. If $\rho: U \longrightarrow \mathbb{R}_$is an (s_2,E,F)-convex function in the fourth sense, then ρ is an (s_1,E,F)-convex function in the fourth sense.

Proof. Since ρ is an (s_2,E,F)-convex function in the fourth sense, one has

$$\rho(tE(u) + (1-t)F(v))$$
$$\leq t^{\frac{1}{s_2}}\rho(E(u)) + (1-t)^{\frac{1}{s_2}}\rho(F(v))$$
$$\leq t^{\frac{1}{s_1}}\rho(E(u)) + (1-t)^{\frac{1}{s_1}}\rho(F(v))$$

for all $u, v \in U$ and $t \in [0,1]$.

Remark 3. For $F = I$, Theorem 2 coincides with Theorem 2.9 in [12].

Next, some properties of composition of (s,E,F)-convex functions in the fourth sense are given.

Theorem 3. Let $\rho_1: \mathbb{R}_+ \longrightarrow \mathbb{R}$ be an increasing function and $\rho_2: U \longrightarrow \mathbb{R}_+$ be a function. If ρ_1 and ρ_2 are two (s,E,F)-convex functions in the fourth sense, then $\rho_1 \circ \rho_2$ is also an (s,E,F)-convex function in the fourth sense.

Proof. Let $u, v \in U$ and $t \in [0,1]$. Then

$$(\rho_1 \circ \rho_2)(tE(u) + (1-t)F(v))$$
$$= \rho_1(\rho_2(tE(u) + (1-t)F(v))$$
$$\leq \rho_1\left(t^{\frac{1}{s}}\rho_2 E(u) + (1-t)^{\frac{1}{s}}\rho_2 F(v)\right)$$
$$\leq \rho_1(t\rho_2 E(u) + (1-t)\rho_2 F(v))$$
$$\leq t^{\frac{1}{s}}\rho_1\left(\rho_2(E(u))\right) + (1-t)^{\frac{1}{s}}\rho_1\left(\rho_2(F(v))\right)$$
$$= t^{\frac{1}{s}}(\rho_1 \circ \rho_2)(E(u)) + (1-t)^{\frac{1}{s}}(\rho_1 \circ \rho_2)(F(v)).$$

Remark 4. For $F = I$, Theorem 3 coincides with Theorem 2.10 in [12].

Theorem 4. If $\rho_1: U \longrightarrow \mathbb{R}$ is an (s,E,F)-convex function in the fourth sense and $\rho_2: \rho_1(U) \longrightarrow \mathbb{R}$ is an increasing linear function, then $\rho_2 \circ \rho_1: U \longrightarrow \mathbb{R}$ is an (s,E,F) convex function in the fourth sense.

Proof. Let $u, v \in U$ and $t \in [0,1]$. Then

$$(\rho_2 \circ \rho_1)(tE(u) + (1-t)F(v))$$
$$= \rho_2\left(\rho_1(tE(u) + (1-t)F(v))\right)$$
$$\leq \rho_2\left(t^{\frac{1}{s}}\rho_1(E(u)) + (1-t)^{\frac{1}{s}}\rho_1(F(v))\right)$$
$$= t^{\frac{1}{s}}\rho_2\left(\rho_1(E(u))\right) + (1-t)^{\frac{1}{s}}\rho_2\left(\rho_1(F(v))\right)$$
$$= t^{\frac{1}{s}}(\rho_2 \circ \rho_1)(E(u)) + (1-t)^{\frac{1}{s}}(\rho_2 \circ \rho_1)(F(v)).$$

Hence, $\rho_2 \circ \rho_1$ is an (s,E,F)-convex function in the fourth sense.

Remark 5. For $F = I$, Theorem 4 coincides with Theorem 2.11 in [12].

Theorem 5. Assume that $\rho_1: U_1 \longrightarrow U_2$ is a linear transformation and $\rho_2: U_2 \longrightarrow \mathbb{R}$ is an (s,E,F)-convex function in the fourth sense, then $\rho_1 \circ \rho_2$ is an (s,E,F)-convex function in the fourth sense.

Proof. Let $u, v \in U$ and $t \in [0,1]$. Then

$$(\rho_1 \circ \rho_2)\big(tE(u) + (1-t)F(v)\big)$$
$$= \rho_1\Big(\rho_2\big(tE(u) + (1-t)F(v)\big)\Big)$$
$$= \rho_1\Big(t\rho_2\big(E(u)\big) + (1-t)\rho_2\big(F(v)\big)\Big)$$
$$\leq t^{\frac{1}{s}}\rho_1\Big(\rho_2\big(E(u)\big)\Big) + (1-t)^{\frac{1}{s}}\rho_1\Big(\rho_2\big(F(v)\big)\Big)$$
$$= t^{\frac{1}{s}}(\rho_1 \circ \rho_2)\big(E(u)\big) + (1-t)^{\frac{1}{s}}(\rho_1 \circ \rho_2)\big(F(v)\big).$$

Remark 6. For $F = I$, Theorem 5 coincides with Theorem 2.12 in [12].

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