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3rd International Conference on Scientific and Academic Research

December 25-26, 2023 : Konya, Turkey

https://www.icsarconf.com/

Existence and Uniqueness Analysis of Solutions in Integral Equations

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Abstract – In this study, proofs of the existence and uniqueness of solutions in integral equations are presented. Based on the hypotheses given depending on the initial conditions and the properties of the equations, the existence of the solution has been demonstrated using Banach fixed point theorem and Lipschitz conditions. Again with the help of inequality techniques some qualitative behaviors of the solutions of the equation and limitation properties of the solutions were examined. With this evidence it has been observed that the contraction map in a complete space always has a fixed point.

Keywords – Banach Fixed Point Theorem, Existence, Integral Equation, Lipschitz Conditions, Inequality Techniques

I. Introduction

Integral equations are global equations because they require integration over the entire space. This means finding the value of the sought function at a point in terms of expressions containing the integral of that function over the entire space. Integral equations are generally much more difficult to solve. For this reason, it is thought that it would be useful to find approximate solutions of this type of systems, which have an important place in the fields of physics and engineering. In recent years, there has been a significant increase in studies on linear and nonlinear integral differential equations [1]- [5]. Nonlinear integral equations are used in many branches of nonlinear functional analysis. The most common areas are; Applications in engineering, mechanics, physics, electrostatics, biology, chemistry and economic theory [6]. The difficult part of these studies is finding solutions when dealing with Volterra-Fredholm fractional integral differential equations. For this reason, many researchers have tried to use different techniques for the solution. Some of the analytical and numerical solutions of these techniques and problems are given in [7]- [9]. In many articles the existence and uniqueness of solutions of some types of integral equations are discussed [10]- [12]. They examined the existence of first-order integral equations and their uniqueness consequences with the argument from type divergence [13]. investigated the existence of solutions to iterative integral differential equations [14]. They demonstrated the existence and uniqueness of the solution for the iterative integral differential.

Based on these problems, in this article we will discuss new existence and uniqueness results for nonlinear variables.

II. Materials and Method

Existence and Uniqueness of the Solution

.To demonstrate the existence and uniqueness of solutions to first-order integral equations, we will first give basic definitions and theorems frequently used in analysis.

Theorem 2.1. (Fredholm Alternative Theorem)

If the homogeneous Fredholm integral equation is as follows,

$$u(x) = \mu \int_a^b f(x,y)u(y)\,dy, \qquad (1)$$

The following formula has only zero solutions, $u(x) = 0$.

Theorem 2.2. (Uniqueness of Solution)

Consider the Fredholm Integral Equation given below,

$$u(x) = h(x) + \mu \int_a^b f(x,y)u(y)\,dy, \quad (2)$$

From this we conclude the following,

1. The kernel $f(x,y)$ in equation (2) is continuous in the square

$$\prod = \{(x,y); a \leq x \leq b, a \leq y \leq b\}$$

2. $f\,(x)$ is a continuous real-valued function,

Then the following condition

$$|\mu|M(b-a)<1 \quad (3)$$

where

$$|f(x,y)| \leq M \in R. \quad (4)$$

These conditions must be met for the existence of a single solution to the equation in (2). Otherwise If the necessary condition in (3) is not satisfied, then a continuous solution may exist.

Definition 2.3. (Metric space)

Let (M,d) denote a metric space. Let M be a set and d be the distance function $d: MxM \to R$ is a distance function. Such that this metric satisfies the following three conditions,

a) For each $x, y \in M$, $0 \leq d(x,y)$ ve $d(x,y) = 0 \Leftrightarrow x = y$.

In other words, the distance between any two elements in the set M can never be negative, and the distance between these two points can be 0 only if the points are equal to each other.

b) For all $x, y \in M$, $d(x,y) = d(y,x)$,

So the distance is symmetric in x and y.

c) For all $x, y, z \in M$, $d(x,z) \leq d(x,y) + d(y,z)$. This inequality is called triangle inequality.

Definition 2.4. (Fixed point of a mapping)

(M,d) metric space, $S \subset M$ and T transformation $T: S \to M, T(u) = v$,
Let it be given wit. The element $u_0 \in S$, where $T(u_0) = u_0$, is called the fixed point of the T transformation.

Definition 2.5. (Shrinkage mapping)
If there is a number $a\ (0 \leq a < 1)$ such that $d(T(u_1),T(u_2)) \leq a\, d(u_1.u_2)$ for $u_1, u_2 \in S$, the function T is called a contraction function.

Definition 2.6. (Lipschitz Condition)
Let f be a function defined on $Q(a,b) \times R$. $\forall\, x, y \in Q(a,b) \times R$ for $|f(x) - f(y)| \leq M|x-y|$ if there is a number M independent of x and y, the function $f(x)$ is said to satisfy the Lipschitz condition on $Q(a,b) \times R$.

Theorem 2.7.(Fixed Point Theorem)
If (M,d) is a complete metric space and $T: M \to M$ is a contraction function, T has one and only one fixed point.

Proof: Let's see that the following conditions are met for the $T(u) = u$ transformation.
a) If there is a fixed point, it is odd.
b) The existence of a fixed point, for this we will first show that the series of consecutive approximations $u_{n+1} = T(u_n)$ is a Cauchy convergent series since it is in the full metric space. We will show that the limit point for the convergent series is $u = \lim_{n\to\infty} u(n)$, $u = T(u)$ is the fixed point of the transformation.
a) Let's assume the opposite to prove that the fixed point is odd. So let's assume that there are points u and v that are different from each other. In this case, $u \neq v$ and hence $d(u,v) \neq 0$.
It is obvious that $u = T(u)$ and $v = T(v)$ and $d(T(u),T(v)) = d(u,v) \neq 0$. Since T is a contraction function, if we use the following three inequalities,

$$d(T(u_1),T(u_2) \leq a\, d(u_1.u_2),$$

$$d\big(T(u),T(v)\big) \leq a\, d(u,v),\ 0 \leq a \leq 1$$

$$d(T(u),T(v)) = d(u,v) \neq 0 \text{ ve } d\big(T(u),T(v)\big) = a\, d(u,v), \quad (5)$$

hence,

$d(u,v) = d\big(T(u),T(v)\big) \leq a\, d(u,v)$

and from here the equality $(1-a)d(u,v) \leq 0$ is reached.

Since $d(u,v) > 0$, $(1-a) \leq 0$ and hence is obtained $a \geq 1$. This contradicts the fact that $0 \leq a < 1$ we stated in the contraction function. This shows that $d(u,v) = 0$, that is $u = v$.

b) To prove the existence of a fixed limit point such that $u = T(u)$, we will show that when we apply iterative operations to the sequence u_n, the sequence $u_{n+1} = T(u_n)$ we obtain is a Cauchy sequence. With the help of the contraction property, we will find the distance $d(u_n, u_{n+1})$ between the first two consecutive approaches $u_1$and u_2 in terms of $d(u_1,u_2)$.

In the next step, we will find the distance $d(u_n, u_{n+p})$. For this, we will use the definition of the Cauchy sequence. (For $\forall \varepsilon > 0$ there is an

$n_0 \in N$, such that $n,m \geq n_0$ if $d(u_n,u_m) < \varepsilon$)

$u_{n+1}(x) = \int F\big(x,y,u_n(y)\big)dy$ from equality,

$d(u_2,u_3) = d(T(u_1,u_2) \leq a\, d(u_1,u_2)$,

if we continue in the same way,

$d(u_3,u_4) = d\big(T(u_2),T(u_3)\big) \leq a\, d(u_2,u_3)$,

Here, for $d(u_2,u_3)$ on the right side of the equation, if we use the inequality

$d(u_2,u_3) \leq a\, d(u_1,u_2)$,

$d(u_3,u_4) \leq a\, d(u_2,u_3) \leq a^2 d(u_1,u_2)$,

if we continue this way

$d(u_n,u_{n+1}) \leq a^{n-1} d(u_1,u_2)$,

the equation is reached. Here, due to the geometric factor a^{n-1}, the distance between the terms u_n and u_{n+1} is smaller than the distance between the terms $u_1$and u_2.

Here we also need to show Cauchy convergence, which will require the use of the previous result and the triangle inequality of the metric $d\big(u_n,u_{n+p}\big)$.

We see that if we use the triangle inequality over and over again, we get

$$d\big(u_n,u_{n+p}\big) \leq d(u_n,u_{n+1}) + d(u_{n+1},u_{n+2}) + d(u_{n+2},u_{n+3}) + \cdots + d\big(u_{n+p-1},u_{n+p}\big).$$

Since $d(u_n,u_{n+1}) \leq a^{n-1} d(u_1,u_2)$ for each of the terms on the right side of this inequality,

$$d\big(u_n,u_{n+p}\big) \leq d(u_n,u_{n+1}) + d(u_{n+1},u_{n+2}) + d(u_{n+2},u_{n+3}) + \cdots + d\big(u_{n+p-1},u_{n+p}\big)$$

$$\leq a^{n-1} d(u_1,u_2) + a^n d(u_1,u_2) + a^{n+1} d(u_1,u_2) + \cdots + a^{n+p-2} d(u_1,u_2)$$

$$= [a^{n-1} + a^n + a^{n-1} + \cdots + a^{n+p-2}] d(u_1,u_2)$$

$$= a^{n-1}(1 + a + a^2 + \cdots + a^{p-1}) d(u_1,u_2)$$

$$= a^{n-1} \frac{1-a^p}{1-a} d(u_1,u_2). \qquad (6)$$

Here, it is clearly seen that the expression in parentheses on the left side of the equation is a geometric series and that the expression on the right side of the equation converges to 0 for $n \to \infty$. This shows that $d\big(u_n,u_{n+p}\big) \to 0$ for $n \to \infty$. Since the sequence u_n is an element of the full metric space, it converges to a point u. Now let's say that this point u is the element of our equation.We need to show that there is a fixed point such that $u = T(u)$. In other words, $d(u,T(u)) = 0$. Hence, $u_{n+1} = T(u_n)$ and the existence of the limit point,

$$\lim_{n\to\infty} u_{n+1} = \lim_{n\to\infty} u(n) = u,$$

or

$n \to \infty$ için $d\big(u,T(u_n)\big) = d(u,u_{n+1}) \to \infty$,

$d(u,u_n) \to 0$. If we use the triangle inequality with these results is obtained

$$d(u,T(u)) \leq d\big(u,T(u_n)\big) + d\big(T(u_n),T(u)\big)$$
$$\leq d\big(u,T(u_n)\big) + a d(u_n,u)$$
$$\leq d(u,u_{n+1}) + a d(u_n,u).$$

After using the distributive property of the T operator in the last term, each term on the right of the equation approaches 0 for $n \to \infty$. This shows that $d(u,T(u)) \leq 0$. Moreover, since the distance d cannot be less than 0, $d(u,T(u)) = 0$, which in the fixed point theorem, it shows that $u = T(u)$.

Existence of Solutions to Linear Fredholm Integral Equations

Let's consider the linear fredholm integral equation,
$u(x) = h(x) + \mu \int_a^b f(x,y)u(y)\,dy$

First we need to find a. Let $h(x)$ be continuous in the range $[a,b]$ and $f(x,y)$, be continuous in the set

$D = \{(x,y): x\epsilon[a.b], y, x\epsilon[a,b]\}$.

For such functions, the continuous full metric space $C[a,b]$ and its metric function $d(x,y)$,

$$d\big(f(x),g(x)\big) = max_{x\epsilon[a,b]}|f(x) - g(x)|; \qquad (7)$$

$f, g\epsilon\ C[a,b]$.

In (7), we must first state that the $f(x,y)$) kernel is limited as a sufficient condition for the $T(u)$ transformation to be a contraction transformation. Since $f(x,y)$ is continuous in the D square region given below, $f(x,y) \leq M$, which means it is limited. To show that T is a contraction transformation, we will use two continuous functions of $C[a,b]$, $\beta(x), \gamma(x)$ and the metric functions $T(\beta(x))$,$T\big(\gamma(x)\big)$.

$$d\Big(T\big(\beta(x)\big), T\big(\gamma(x)\big)\Big) = max_{x\epsilon[a,b]}\Big|g(x) + \mu\int_a^b f(x,y)\beta(y)dy - [g(x) + \mu\int_a^b f(x,y)\gamma(y)\,dy\Big|$$
$$= max_{x\epsilon[a,b]}\Big|\mu\int_a^b f(x,y)[\beta(y) - \gamma(y)dy]\Big|$$
$$\leq max_{x\epsilon[a,b]}\int_a^b |\mu f(x,y)[\beta(y) - \gamma(y)]|\,dy$$
$$\leq |\mu| M max_{x\epsilon[a,b]}\int_a^b |\beta(y) - \gamma(y)]|\,dy$$
$$\leq |\mu| M(b-a) d(\beta(x), \gamma(x) = \ \alpha d(\beta(x), \gamma(x).$$

Since the upper limit of the $f(x,y)$ function is M, here

$$\alpha = |\mu M(b-a)| < 1. \qquad (8)$$

Since the linear fredholm integral equation is $\alpha < 1 (\mu < \frac{1}{M(b-a)})$, in the equation $\alpha = \mu M(b-a)$ in (8), it becomes a shrinkage function. In this case, the u_n input has the maximum error limit. It will produce the output u_{n+1}.

$$\varepsilon = max_{x\epsilon[a,b]}|u - u_{n+1}\Big| \leq \frac{|\mu M(b-a)|^n}{1 - |\mu M(b-a)|} max_{x\epsilon[a,b]}\Big|u_2 - u_1|, \qquad (9)$$

Here, the estimated maximum difference between the first two terms $u_1$and u_2 is $\mu|M(b-a)| < 1$.

Existence of Solutions to Nonlinear Fredholm Integral Equations

Let the nonlinear Fredholm integral equation be given as follows,

$$u(x) = h(x) + \mu \int_a^b F(x,y,u(y))\,dy$$

The existence of a solution to this equation is possible if the following conditions are met:

1. For $a \leq x \leq b$, the function $h(x)$ is limited, that is, $h(x) < R$.

2. For $a \leq x, y \leq b$, the function $F\big(x,y,u(y)\big)$ is integrable and bounded. So $\big|F\big(x,y,u(y)\big)\big| < M$.

3. The function $F\big(x,y,u(y)\big)$) satisfies the Lipschitz condition. That is, $|F(x,y,z_1) - F(x,y,z_2)| < M|z_1 - z_2|$.

Theorem 2.8. Nonlinear fredholm integral equation be given as follows,

$u(x) = h(x) + \mu \int_a^b F(x,y,u(y))\,dy$

Also, let's assume that the function $F(x,y,z)$ is defined on the set $Q(a,b) \times R$ and satisfies the Lipschitz conditions. That is,

$|F(x,y,z_1) - F(x,y,z_2)| < M|z_1 - z_2|$.

Then, the given integral equation is for $\mu < \frac{1}{M(b-a)}$ on $[a,b]$ it has only one solution.

III. Results

In integral equations, the existence and uniqueness of solutions are investigated depending on a particular problem or type of equation. The existence and uniqueness of the solution in an integral equation are shaped by the initial conditions and the properties of the equations. Here, this examination for the Fredholm equation was made with the fixed point theorem.

IV. DISCUSSION

If an integral equation has a solution, that solution is usually defined on a certain interval and the definite integral must be taken. Uniqueness means that the solution of an equation is sufficiently regular within a certain range. In integral equations, the uniqueness of solutions is examined with Lipschitz conditions or special theorem.

V. CONCLUSION

The existence and uniqueness of solutions of integral equations are demonstrated through various theorems and methods used in mathematics. The existence and uniqueness of solutions can be proven for different integral equations using different techniques and approaches. We demonstrated the existence and uniqueness of the solution for the Fredholm equation by using the fixed point theorem, the Lipschitz condition, and some inequalities. Banach's Fixed Point Theorem states that a contraction map in a full space always has a fixed point. It is important in making predictions for a specific integral equation in the mathematical literaturetion.

ACKNOWLEDGMENT

Not applicable.

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