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Construction of Green's Functions when the Boundary Value Problem Contains a Parameter

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Abstract – This article provides a general framework for various differential operators and boundary value problems, focusing on establishing parameter-containing Green's functions. The study investigates how Green's functions varying within a parameter λ contribute to the solution of integral equations. The focus of the study is on how parameterized Green's functions can be derived using analytical and numerical methods and how these functions can be related to specific differential equations.

Keywords – Green's Function, Boundary Value Problems, Discontinuity, Parameter, Eigenvalue.

I. Introduction

One of the important models of mathematical physics is problems involving differential equations. The concept of Green's functions is also important among the problems that are frequently used in applied sciences. Green's functions are also used to examine the properties of excitation spectra through an analytical approach in the complex energy plane. Magnetic of solids Its susceptibility, electrical conductivity and interactions with the applied external field can also be expressed with Green's functions. The most important feature of Green's functions is that they are time dependent. Using these functions, you can determine whether solids are isotropic or anisotropic. Its properties can be examined depending on time. The Green's function method is especially useful in explaining the physical properties of multi-particle systems that interact with each other. Theory of potential Green's functions; It emerged to determine the electromagnetic fields that occur due to resource insufficiency, that is, produced by load and current loss. There are many studies on examining the spectral properties of the differential operator. Analytically, for some linear problems involving equations with uniform coefficients under linear conditions, Green's function has been obtained [1]-[3]. Second method, it is the classical Green's function method [4]. Part of the problem addressed by this method an attempt is made to establish a classical equivalent problem with an equality through integration. For such a problem, by the second method,analytically Green or generalized Green, although it exists under certain conditions A clear expression of the function may not be obtained. The third method is; basicall for problems involving discontinuities based on the method of change of parameters. It is an improved basic solution method. There are some problems with this method Green's function was created for [5]-[7]. Studies on the asymptotic state of the eigenvalues of differential operators under different boundary conditions and the problems of establishing the Green's functions are given in [8]-[12]. In this study, the structure and establishment of the Green's functions encountered in obtaining the solutions of initial value and boundary value problems related to differential equations will be discussed.

II. Materials and Method

Establishing the Green's function

In this study, the boundary value problem containing parameters and also the we will be interested in finding the asymptotic expression of the eigenvalues of the operator.

Let's look at the following differential operator, which is frequently encountered in many problems of physics.
Let's define the differential operator L, which is encountered in many problems of mathematical physics, as follows,

$$L: y''(x) + \{\lambda - f(x)\}y(x) = 0 \tag{1}$$

$$\lambda y(0) = -y'(0), \tag{2}$$

$$(\alpha_1 + \lambda\widetilde{\alpha_1})\, y(\pi) = (\alpha_2 + \lambda\widetilde{\alpha_2})\, y'(\pi) \tag{3}$$

$q(x) \in W_2^1[0,1]$ a complex valued function, α_1, α_2 complex number and λ, spectral is the parameter. Let the two solutions of the boundary value problem (1)-(3) be given with the following conditions,

$$\emptyset(x,\lambda);\ \emptyset(0,\lambda) = 1\ \emptyset'(0,\lambda) = 0,$$

$$\varphi(x,\lambda);\ \emptyset(0,\lambda) = 0, \emptyset'(0,\lambda) = 1. \tag{4}$$

$$W\{\emptyset(x,\lambda), \varphi(x,\lambda)\} = \begin{vmatrix} \emptyset(x,\lambda) & \varphi(x,\lambda) \\ \emptyset'(x,\lambda) & \varphi'(x,\lambda) \end{vmatrix} = \emptyset(x,\lambda)\varphi'(x,\lambda) - \emptyset'(x,\lambda)\varphi(x,\lambda)$$

$$\frac{d}{d\lambda} W\{\emptyset(x,\lambda), \varphi(x,\lambda)\} = 0.$$

obtained. In this situation $W\{\emptyset(x,\lambda), \varphi(x,\lambda)\}$ is constant. If (4) conditions are used

$$W\{\emptyset(x,\lambda), \varphi(x,\lambda)\} = \begin{vmatrix} \emptyset(0,\lambda) & \varphi(0,\lambda) \\ \emptyset'(0,\lambda) & \varphi'(0,\lambda) \end{vmatrix} = 1 \neq 0,$$

then these two functions are linearly independent.

C_1 and C_2 the general solution of equation (1) with arbitrary constants,

$$y(x,\lambda) = C_1\emptyset(x,\lambda) + C_2\varphi(x,\lambda)$$

Using the variation of constant method

$$-y''(x) + q(x)y(x) = \lambda y(x) - f(x)$$

Let's get the general solution of the equation as follows

$$y(x,\lambda) = C_1(x,\lambda)\emptyset(x,\lambda) + C_2(x,\lambda)\varphi(x,\lambda) \tag{5}$$

$C_1(x,\lambda)$ and $C_2(x,\lambda)$ we can choose functions such that they satisfy the following equations

$$C_1'(x,\lambda)\emptyset(x,\lambda) + C_2'(x,\lambda)\varphi(x,\lambda) = 0$$

$$C_1'(x,\lambda)\emptyset(x,\lambda) + C_2'(x,\lambda)\varphi(x,\lambda) = f(x).$$

It is obtained from this system that

$$C_1'(x,\lambda) = \frac{-\varphi(x,\lambda)f(x)}{W(\lambda)}$$

$$C_2'(x,\lambda) = \frac{\emptyset(x,\lambda)f(x)}{W(\lambda)}$$

$$C_1(x,\lambda) = -\frac{1}{W(\lambda)}\int_x^\pi \varphi(u)f(u)\,du + C_1(\lambda)$$

$$C_2(x,\lambda) = \frac{1}{W(\lambda)}\int_x^\pi \emptyset(u)f(u)\,du + C_2(\lambda).$$

This is what we found If we replace the expressions in (5),

$$y(x,\lambda) = \emptyset(x,\lambda)\left[-\frac{1}{W(\lambda)}\int_x^\pi \varphi(u)f(u)\,du + C_1(\lambda)\right] + \varphi(x,\lambda)\left[\frac{1}{W(\lambda)}\int_x^\pi \emptyset(u)f(u)\,du + C_2(\lambda)\right]$$

$$y(x,\lambda) = -\frac{\emptyset(x,\lambda)}{W(\lambda)}\left[\int_x^\pi \varphi(u)f(u)\,du\right] + \frac{\varphi(x,\lambda)}{W(\lambda)}\left[\int_x^\pi \emptyset(u)f(u)\,du\right] + C_1(\lambda)\emptyset(x) + C_2(\lambda)\varphi(x) \tag{6}$$

$$y'(x,\lambda) = \frac{1}{W(\lambda)}\left[\emptyset'(x,\lambda)\int_x^\pi \varphi(u)f(u)\,du - \emptyset(x,\lambda)\varphi(x)f(x)\right] + \frac{1}{W(\lambda)}\left[\varphi'(x,\lambda)\int_x^\pi \emptyset(u)f(u)\,du + \emptyset(x,\lambda)\varphi(x)f(x)\right] + C_1(\lambda)\emptyset'(x) + C_2(\lambda)\varphi'(x) \tag{7}$$

obtained. If the expressions (6) and (7) are replaced in the condition (1),

$$\lambda\left[\frac{\emptyset(0)}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda)\emptyset(0) + C_2(\lambda)\varphi(0)\right]$$

$=-\left[\frac{\emptyset'^{(0)}}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda)\varphi'^{(0)} + C_2(\lambda)\varphi'(0)\right]$

If equations (6) are used here

$$\lambda\left[-\frac{1}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda) + C_2(\lambda)\varphi(0)\right]$$

$$=-\left[\frac{\lambda}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + \lambda C_1(\lambda) + C_2(\lambda)\varphi'(0)\right],$$

and from here

$$\lambda C_1(\lambda) + \lambda C_2(\lambda)\varphi(0) = -\lambda C_1(\lambda) + C_2(\lambda)\varphi'(0)$$

$$[\lambda\varphi(0) + \varphi'(0)]C_2(\lambda) = 0$$

$$\lambda\varphi(0) + \varphi'(0) = -W(\lambda) \neq 0,$$

is $C_2(\lambda) = 0$.

Similarly, if the expressions (6) and (7) are replaced in the condition (3),

$$(\alpha_1 + \lambda\widetilde{\alpha_1})\left[\frac{\emptyset(\pi)}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda)\emptyset(\pi) + C_2(\lambda)\varphi(\pi)\right]$$

$$=(\alpha_2 + \lambda\widetilde{\alpha_2})\left[\frac{\emptyset\prime(\pi)}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda)\emptyset'(\pi) + C_2(\lambda)\varphi'(\pi)\right]$$

It is possible. If equations (6) are used,

$$(\alpha_1 + \lambda\widetilde{\alpha_1})\left[\frac{(\alpha_2 + \lambda\widetilde{\alpha_2})}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda)\emptyset(\pi) + C_2(\lambda)(\alpha_2 + \lambda\widetilde{\alpha_2})\right]$$

$$=(\alpha_2 + \lambda\widetilde{\alpha_2})\left[\frac{(\alpha_1+\lambda\widetilde{\alpha_1})}{W(\lambda)}\int_0^\pi \varphi(u)f(u)\,du + C_1(\lambda)\emptyset'(\pi) + C_2(\lambda)(\alpha_1 + \lambda\widetilde{\alpha_1})\right]$$

$$(\alpha_1 + \lambda\widetilde{\alpha_1})C_1(\lambda)\emptyset(\pi) + (\alpha_1 + \lambda\widetilde{\alpha_1})C_2(\lambda)(\alpha_2 + \lambda\widetilde{\alpha_2})$$

$$=(\alpha_2 + \lambda\widetilde{\alpha_2})C_1(\lambda)\emptyset'(\pi) + (\alpha_2 + \lambda\widetilde{\alpha_2})\,C_2(\lambda)(\alpha_1 + \lambda\widetilde{\alpha_1}),$$

from here

$$[(\alpha_1 + \lambda\widetilde{\alpha_1})\emptyset(\pi) - (\alpha_2 + \lambda\widetilde{\alpha_2})\emptyset'(\pi)]C_1(\lambda) = 0$$

$$(\alpha_1 + \lambda\widetilde{\alpha_1})\emptyset(\pi) - (\alpha_2 + \lambda\widetilde{\alpha_2})\emptyset'(\pi) = W(\lambda) \neq 0,$$

is $C_1(\lambda) = 0$.

Then, for the solution of equation (1) that satisfies the conditions of (2)-(3),

$$y(x,\lambda) = \frac{\emptyset(x)}{W(\lambda)}\left[\int_x^\pi \varphi(u)f(u)\,du\right] + \frac{\varphi(x)}{W(\lambda)}\left[\int_0^x \emptyset(u)f(u)\,du\right] \tag{8}$$

formula is found here

$$G(x,u,\lambda) = \begin{cases} \dfrac{\varphi(x)\emptyset(u)}{W(\lambda)}, & 0 \le u \le x \le \pi \\ \dfrac{\emptyset(x)\varphi(u)}{W(\lambda)}, & 0 \le x \le u \le \pi \end{cases} \tag{9}$$

If we take the equation (8) as it is written as. In this way, the Green's function of the boundary value problem (1)-(3) becomes (9).

A Limit Value Depending On The Parameter Inverse Problem For Problem

$$-y''(x) + f(x)y(x) = \lambda\tau(x)y \tag{10}$$

where

$$\tau(x) = \begin{cases} 1, & 0 \le x < a \\ \delta^2, & a \le x \le \pi \end{cases}, \quad 0 < a \neq 1$$

is satisfied. Let it be given with (2) initial conditions. The following integral representation is available for the solution of the equation

$$e(x,\lambda) = e_0(x,\lambda) + \int_{\boldsymbol{\mu}^-(x)}^{\boldsymbol{\mu}^+(x)} K(x,t)e^{i\lambda t}dt,$$

here

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda t}, & 0 \le x < a, \\ \frac{1}{2}\left(1 + \frac{1}{\sqrt{\tau(x)}}\right)e^{i\lambda\boldsymbol{\mu}^+(x)} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{\tau(x)}}\right)e^{i\lambda\boldsymbol{\mu}^-(x)}, & a \le x \le \pi, \end{cases}$$

It is $K(x,.) \in L_1$,

$\mu^{\mp}(x) = \mp \sqrt{\tau(x)} + a\left(1 \mp \sqrt{\tau(x)}\right).$

Moreover, K_x' variant is available and provides the following features,

$$\frac{d}{dx}\{K(x,\mu^-(x)+0) - K(x,\mu^+(x)-0)\} = \frac{1}{\sqrt[4]{\tau(x)}}(1 - \frac{1}{\sqrt{\tau(x)}})\, f(x),$$

$$K(x, -\mu^+(x)) = 0.$$

In addition to these features, $f(x)$ is a differentiable function. In this case, the following features also apply,

$$\tau(x)K_{tt}'' - K_{xx}'' + f(x)K = 0, \quad |t| < \mu^+(x),$$

$$\int_{\boldsymbol{\mu}^-(\boldsymbol{x})}^{\boldsymbol{\mu}^+(\boldsymbol{x})} |K(x,t)|\, dt \leq C\left(exp\left\{\int_0^x |f(t)|\, dt\right\} - 1\right), \quad 0 < C = \text{stable}.$$

Corollary. Special solution of equation (10) that satisfies the conditions of (2)

$$\psi(\boldsymbol{x},\boldsymbol{\lambda}) = \psi_0(x,\lambda) + \int_{\boldsymbol{0}}^{\boldsymbol{\mu}^+(x)} A(x,t)\frac{sin\lambda t}{\lambda} dt$$

is of the form, here $A(x,t)$ kernel

$K(x,t) = K(x,-t)$ realizes its properties.

III. Results

In this study, the establishment of Green's function, which is encountered in obtaining solutions to initial value and boundary value problems related to differential equations, is discussed. For this purpose, Green's function for initial value problems related to differential equations has been examined.

IV. DISCUSSION

Even in the case of a simple linear differential equation and a nonlocal condition, the problem under consideration may not have a meaningful classical type equivalent problem. For these reasons, some serious difficulties may arise in using the classical methods mentioned for such a problem. Similar difficulties may arise even in problems of the classical type where the coefficients of the differential equation are non-differentiable continuous functions. In order to contribute to the elimination of these difficulties, the concepts of Green's and generalized Green's functionals have been introduced and solutions to these problems have been investigated within this framework.

V. CONCLUSION

The analytical nature of parameter-containing linear differential operators was examined, and the construction method of the Green's function of parameter-containing differential operators was demonstrated by the application of the theory of functions of complex variables. The remarkable feature of this study is that both the equation and the boundary conditions, there is an eigenvalue parameter. It can also be applied in finding analytical and approximate solutions to many concrete problems in physics.

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