All Sciences Proceedings
http://as-proceeding.com/

2nd International Conference on Contemporary Academic Research
November 4-5, 2023 : Konya, Turkey

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Structure and Solution of Integral Equations with Fixed Kernels

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Abstract **–** The aim of this study is to use the Fredholm integral. To examine equations and methods of finding exact solutions. Finding exact solutions for linear and nonlinear Fredholm is also important in the scientific arena. At this point, we applied the sequential approximations method to find a numerical solution for a special type of Fredholm, using the kernel function. comparison of integral equation results with the exact solution. It was observed that the results obtained were close to the final results. Thus, the effectiveness and simplicity of the method has been proven.

Keywords –, Fredholm Systems, Integral, Sequential Approximations, Solution, Kernel

I. Introduction

Integral equations in short; It can be defined as equations in which the unknown function is under the integral sign. However, this definition is inadequate. Because, based on this definition, it is not possible to establish a theory that will cover all integral equations. Therefore, it would be correct to evaluate each integral equation on its own. Thus, a very wide research field is opened and the subject requires detailed examination. In terms of basic concepts, integral equations are primarily divided into two large classes as linear and non-linear integral equations. This classification varies depending on whether the integral operator is linear or not with respect to the u(x) operator. In this section, we will focus on the structure of Fredholm integral equations, regardless of whether the integral equations are linear or not. Regardless of whether they are linear and homogeneous,

$$\phi(x) = \int_a^b K(x,t)\,u(t)dt\,,$$

$$u(x) = \int_a^b K(x,t)\,u(t)dt\,,$$

$$u(x) = f(x) + \int_a^b K(x,t)\,u(t)dt\,,$$

$$\phi(x)u(x) = f(x) + \int_a^b K(x,t)\,u(t)dt\,,$$

Equations in the form of which the lower and upper limits are equal to a constant value are called Fredholm integral equations.To date, many methods have been used to solve integral equations.scientific approach methods were applied. To solve the Fredholm integral equation of the second kind B-spline wavelet method, Method of moments based on B-spline wavelets Maleknejad and Sahlan [1] and variational iteration method (VIM) [2], [3] was applied. It has been applied to solve the nonlinear Fredholm integral equation. Some researchers have proposed some numerical methods for Fredholm linear integral equations. Some of these are Rationalized Haar functions methods, Taylor series expansion method [4]–[6], Haar Wavelet method with operational integration matrices [7] Apart from these, for quadratic equations Squaring method [8], B-spline wavelet method [9], wavelet Galerkin method [10]. In addition to these, Homotopy perturbation method (HPM) [11]–[13] and Adomian decomposition method (ADM) [14], [15]. It has been applied to solve the nonlinear Fredholm integral equation.

II. MATERIALS AND METHOD

A) Integral Equatıons Wıth Fıxed Nuclear

When searching for solutions to Fredholm integral equations as follows, the type of $K(x,t)$nucleus plays a decisive role

$$\phi(x) = \int_a^b K(x,t)\, u(t) dt \,. \quad (1)$$

Let's assume that the function $K(x,t)$ given in the form is constant. In this case, the Fredholm integral equation is we can express it like this,

$$\phi(x) = f(x) + \lambda \int_a^b c\phi(t)\, dt \,,$$

$$\phi(x) = f(x) + \lambda c \int_a^b \phi(t)\, dt \,.$$

Here, if $\lambda c = \mu$ is taken as

$$\phi(x) = f(x) + \mu \int_a^b \phi(t)\, dt \,,$$

can be written as. Moreover, since the expression $\int_a^b \phi(t)\, dt$ dt is a finite value, if we denote this value with A, hen the integral equation it is written like this,

$$\phi(x) = f(x) + \mu A. \quad (2)$$

Since this solution satisfies the integral equation, the following equation is valid,

$$f(x) + \mu A = f(x) + \mu \int_a^b \{f(t) + \mu A\}\, dt.$$

If we edit here,

$$A = \int_a^b f(t)\, dt + \mu \int_a^b A dt \,,$$

$$A[1\text{-}\mu \int_a^b dt] = \int_a^b f(t)\, dt \,,$$

or

$$A = \frac{1}{1-\mu(b-a)} \int_a^b f(t)\, dt \quad (3)$$

is available. $1 - \mu(b-a) \neq 0$ and $f(x)$ function are also known, if the value found for A is substituted in the equation $\phi(x) = f(x) + \mu A$.

$$\phi(x) = f(x) + \mu[\frac{1}{1-\mu(b-a)} \int_a^b f(t)\, dt] \,,$$

Also, if the value $\mu = \lambda c$ is written instead,

$$\phi(x) = f(x) + \frac{\lambda c}{1-\lambda c(b-a)} \int_a^b f(t)\, dt. \quad (4)$$

Since the right side of this equation consists of known values, the value of $\phi(x)$ can be found directly.

*Example 1.*Integral equation with fixed kernel $\phi(x) = sinx + \int_0^1 2\phi(t)\, dt$ let it be. Let's find the solution to this, if $A = \int_0^1 \phi(t)\, dt$ is taken as dt, $\phi(x) = sinx + 2A$. If this expression is substituted into the equation,

$$sinx + 2A = sinx + \int_0^1 (sint + 2A) dt \,,$$

$$A = cos1 - 1 \,,$$

is available. Thus, the integral equation solution is

$$\phi(x) = sinx + 2(cos1 - 1).$$

B)Integral Equatıons Wıth Degenerate Nuclear

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)\, dt, \quad (5)$$

$K(x,t)$ kernel in the integral equation if $K(x,t) = r(x)\mathrm{s(t)}$ this kernel is called degenerate kernel and in this case the integral equation is

$$\phi(x) = f(x) + \lambda \int_a^b r(x)\mathrm{s(t)}\phi(t)\, dt.$$

Since $r(x)$ is independent of *t*,

$$\phi(x) = f(x) + \lambda r(x) \int_a^b \mathrm{s(t)}\phi(t)\, dt \,,$$

Here A is the constant value of the integral.

$$A = \int_a^b \mathrm{s(t)}\phi(t)\, dt \,, \quad (6)$$

taken as

$$\phi(x) = f(x) + \lambda A r(x) \,, \quad (7)$$

can be written as . Since this expression is a solution to the integral equation

$$f(x) + \lambda A r(x) = f(x) + \lambda r(x) \int_a^b \{f(t) + \lambda A r(t)\}\mathrm{s(t)}\, dt \,,$$

$$A = \int_a^b \{f(t) + \lambda A r(t)\}\mathrm{s(t)}\, dt \,,$$

$$A = \int_a^b f(t)\mathrm{s(t)}\, dt + \lambda A \int_a^b r(t)\mathrm{s(t)}\, dt \,,$$

$$A\left(1 - \lambda \int_a^b r(t)\mathrm{s(t)}\, dt\right) = \int_a^b f(t)\mathrm{s(t)}\, dt \,,$$

$$A = \frac{\int_a^b f(t)\mathrm{s}(\mathrm{t})dt}{1-\lambda\int_a^b r(t)\mathrm{s}(\mathrm{t})dt}. \quad (8)$$

equality is reached [16]. Since $f(x), r(x)$ and $s(x)$ on the right side of this equation are known functions, in this case $\phi(x)$ is the solution function

$$\phi(x) = f(x) + \frac{\lambda\int_a^b f(t)\mathrm{s}(\mathrm{t})dt}{1-\lambda\int_a^b r(t)\mathrm{s}(\mathrm{t})dt}\, r(x). \quad (9)$$

*Example 2.*Let's solve the integral equation $\phi(x) = \mathrm{e}^x + \int_0^1 \mathrm{e}^{x+\mathrm{t}}\,\phi(t)dt$. Kernel function in the given integral equation

$\mathrm{K}(x,t) = \mathrm{e}^{x+\mathrm{t}} = \mathrm{e}^x\mathrm{e}^{\mathrm{t}}$ It is in the form. Thus, we can express the kernel function as a degenerate kernel.

Since $r(x)s(t) = \mathrm{e}^x\mathrm{e}^{\mathrm{t}}$, it can be written as $r(x) = \mathrm{e}^x$ and $s(t) = \mathrm{e}^{\mathrm{t}}$. Then the integral equation

$$\phi(x) = \mathrm{e}^x + \mathrm{e}^x\int_0^1 \mathrm{e}^{\mathrm{t}}\,\phi(t)dt.$$

If it is written as $\mathrm{A} = \int_0^1 \mathrm{e}^{\mathrm{t}}\,\phi(t)dt$, it can be written as $\phi(x) = \mathrm{e}^x + A\mathrm{e}^x$. If this solution is substituted into the equation

$$\mathrm{e}^x + \mathrm{e}^x A = \mathrm{e}^x + \mathrm{e}^x\int_0^1 \mathrm{e}^{\mathrm{t}}\{\mathrm{e}^t + \mathrm{e}^t A\}dt\ ,$$

$$A\left(1 - \int_0^1 \mathrm{e}^{2\mathrm{t}}\,dt\right) = \int_0^1 \mathrm{e}^{2\mathrm{t}}\,dt\ ,$$

If $A = \frac{e^2-1}{3-e^2}$ is written here obtained

C)Successive Approximations Method

Let's consider the integral equation.

$$u(x) = f(x) + \lambda\int_a^b K(x,y)\,u(y)dy. \quad (10)$$

First, if we take $\lambda = 0$, the integral equation becomes $u(x) = f(x)$. If we write $u(x)$ instead $u_0(x)$ as the first notation here and similarly use the same notation

$$u_1(x) = f(x) + \lambda\int_a^b K(x,y)\,u_0(y)dy\ , \quad (11)$$

obtained. If we express the integral here as follows,

$$\phi_1(x) = \int_a^b K(x,y)\,u_0(y)dy,$$

equation (11) is written as

$$u_1(x) = f(x) + \lambda\phi_1(x). \quad (12)$$

Similarly if we write $u_0(x) = f(x) = \phi_0(x)$) and replace this equation in (12),

$$u_1(x) = \phi_0(x) + \lambda\phi_1(x), \quad (13)$$

we obtain. Using equation (11), we can write $u_2(x)$ as follows,

$$u_2(x) = f(x) + \lambda\int_a^b K(x,y)\,u_1(y)dy. \quad (14)$$

Then, if we replace equation (13) in (14), we get

$$u_2(x) = f(x) + \lambda\int_a^b K(x,y)\,[\phi_0(y) + \lambda\phi_1(y)]dy\ ,$$

$$u_2(x) = f(x) + \lambda\int_a^b K(x,y)\phi_0(y)dy + \lambda^2\int_a^b K(x,y)\phi_1(y)dy.$$

With the actions we will take from here and the following equations

$$\phi_1(x) = \int_a^b K(x,y)\,\phi_0(y)dy,$$

$$\phi_2(x) = \int_a^b K(x,y)\,\phi_1(y)dy\ ,$$

$$u_2(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(t)\ ,$$

is found. Continuing in this way, the function sequence $u_0(x)$,$u_1(x)$,$u_2(x)$,...,$u_{n-1}(x)$,$u_n(x)$and $u_n(x) = \phi_0(x) + \lambda\phi_1(x) + \lambda^2\phi_2(t) + \cdots + \lambda^{\mathrm{n}}\phi_{\mathrm{n}}(x)$ series is obtained. for n=1,2,3,… and so $\phi_{\mathrm{n}}(x)$ is found as follows,

$$\phi_0(x) = f(x) \text{and } \phi_{\mathrm{n}}(x) = \int_a^b K(x,y)\phi_{\mathrm{n}-1}(y)dy.$$

Thus, by the method of successive approximations any real valued solution function is obtained with $\phi_{\mathrm{n}}(x)$.

III. Results

Finding Fredholm integral equations and their exact solutions different studies have been carried out about the methods until today. Exact analysis of linear and non-linear Fredholm equations one of the effective methods to find the results is the method of successive approximations.To examine the convergence between the numerical solution and the exact solution the method is applied to find the numerical solution of a Fredholm Equation.

IV. DISCUSSION

A system of algebraic equations is obtained with the methods generally applied to the Fredholm integral, and this system can now be easily solved. Another way to solve integral equations instead of using a solver is considered here, the degenerate kernel. The solvability of integral equations in various cases has been analyzed using the successive approximation method to consider an important class of kernels.

V. CONCLUSION

In many physical, engineering and electrical phenomena, solutions can be found with integral equations that represent the main equation. A typical example of such an equation is the Fredholm integral. The Fredholm integral equation is often studied as an inverse problem. Many currently available techniques make strong assumptions about its regularity. Here, the structure and properties of the integral are examined if it contains a constant core. A numerical solution can also be found using the sequential approximations technique. The results obtained with this method overlap with the numerical results obtained with the exact solution and there is convergence.

ACKNOWLEDGMENT

Not applicable.

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