All Sciences Proceedings
http://as-proceeding.com/

2nd International Conference on Contemporary Academic Research
November 4-5, 2023 : Konya, Turkey

ICCAR
https://as-proceeding.com/index.php/iccar

On Exponential Type Multiplicatively (s, P)-Functions

Serap Özcan

Department of Mathematics, Kırklareli University, Turkey

serapozcan87@gmail.com

Abstract – In this paper, the concept of exponential type multiplicatively (s, P)-function is introduced. Some algebraic properties of the newly defined classes of functions are studied. Integral inequalities of Hermite-Hadamard type are established for the novel classes of functions in the setting of multiplicative calculus. Also, Hermite–Hadamard type inequalities are derived for the multiplication and division of exponential type multiplicatively (s, P)-functions. The findings of this work may stimulate further research for the researchers working in this field.

Keywords – Exponential Type Multiplicatively Convexity, Multiplicative Convexity, Multiplicative Calculus, Hermite-Hadamard's Inequality

I. Introduction

The function $f: \mathfrak{J} \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(\omega\varkappa + (1-\omega)y) \leq \omega f(\varkappa) + (1-\omega)f(y)$$

holds for all $\varkappa, y \in \mathfrak{J}$ and $\omega \in [0,1]$.

The function f is said to be concave if $-f$ is convex.

The most renowned inequality associated with the integral mean of a convex function is Hermite-Hadamard's inequality, which is given as follows (see [1,2]):

Theorem 1. Let $f: \mathfrak{J} \to \mathbb{R}$ be an integrable convex function. Then

$$f\left(\frac{\varkappa + y}{2}\right) \leq \frac{1}{y-\varkappa}\int_{\varkappa}^{y} f(\alpha)d\alpha \leq \frac{f(\varkappa) + f(y)}{2}.$$

In recent years, Hermite-Hadamard's inequality has garnered renewed attention and have become a significant tool for mathematical analysis, probability theory, optimization, and other branches of mathematics. Furthermore, through examining diverse scenarios, this inequality has been discovered to possess a multitude of applications. For various extensions and generalizations of the Hermite-Hadamard's inequality utilizing innovative and novel approaches, refer to [4–14] and the respective references mentioned therein.

Now, we will give some basic definitions and results.

Definition 1. [2] A function $f: \mathfrak{J} \to (0, \infty)$ is said to be logarithmically or multiplicatively convex on set ꞁ, if

$$f(\varkappa + \omega(y-\varkappa)) \leq (f(\varkappa))^{1-\omega} \cdot (f(y))^{\omega}$$

for all $\varkappa, y \in \mathfrak{J}$ and $\omega \in [0,1]$.

In [3], Ali et al. established Hermite-Hadamard inequality for multiplicatively convex functions as follows:

Theorem 2. Let f be a positive and multiplicatively convex function on interval $[\varkappa, y]$. Then

$$f\left(\frac{\varkappa+y}{2}\right) \leq \left(\int_{\varkappa}^{y} (f(\alpha))^{d\alpha}\right)^{\frac{1}{y-\varkappa}} \leq G(f(\varkappa), f(y))$$

where $G(.,.)$ is the geometric mean.

In [8], Ozcan introduced the class of exponential type multiplicatively convex functions and established Hermite-Hadamard inequality for this class of functions as follows:

Definition 2. A positive function $f: \mathfrak{I} \to \mathbb{R}$ is called exponential type multiplicatively convex if,

$$f(\omega\varkappa + (1-\omega)y) \leq (f(\varkappa))^{e^{\omega}-1}(f(y))^{e^{1-\omega}-1}$$

for all $\varkappa, y \in I$ and $\omega \in [0,1]$.

Theorem 3. Let f be an exponential type multiplicatively convex function on $[\varkappa, y]$. Then

$$f\left(\frac{\varkappa+y}{2}\right) \leq \left(\int_{\varkappa}^{y} f(\alpha)^{d\alpha}\right)^{\frac{2(\sqrt{e}-1)}{y-\varkappa}} \leq [f(\varkappa)f(y)]^{2(\sqrt{e}-1)(e-2)}$$

II. Preliminaries

A. *Multiplicative Calculus*

Recall that the concept of multiplicative integral or * integral is denoted by $\int_{\varkappa}^{y}(f(\alpha))^{d\alpha}$ while the ordinary integral is denoted by $\int_{x}^{y}(f(\alpha))d\alpha$. This is because the sum of the terms of product is used in the definition of a classical Riemann integral of f on $[\varkappa, y]$, the product of terms raised to certain powers is used in the definition of * integral of f on $[\varkappa, y]$.

There is the following relation between Riemann integral and * integral ([15]):

Proposition 1. If f is positive and Riemann integrable on $[\varkappa, y]$, then f is * integrable on $[\varkappa, y]$ and

$$\int_{\varkappa}^{y} (f(\alpha))^{d\alpha} = e^{\int_{\varkappa}^{y} \ln(f(\alpha))d\alpha}$$

In [5], Bashirov et al. show that * integral has the following results and notations:

Proposition 2. If f is positive and Riemann integrable on $[\varkappa, y]$, then f is multiplicatively integrable on $[\varkappa, y]$ and

I. $\int_{\varkappa}^{y}((f(\alpha))^{p})^{d\alpha} = \int_{\varkappa}^{y}((f(\alpha))^{d\alpha})^{p}$,

II. $\int_{\varkappa}^{y}(f(\alpha)g(\alpha))^{d\alpha} = \int_{\varkappa}^{y}(f(\alpha))^{d\alpha} \cdot \int_{\varkappa}^{y}(g(\alpha))^{d\alpha}$,

III. $\int_{\varkappa}^{y}\left(\frac{f(\alpha)}{g(\alpha)}\right)^{d\alpha} = \frac{\int_{\varkappa}^{y}(f(\alpha))^{d\alpha}}{\int_{\varkappa}^{y}(g(\alpha))^{d\alpha}}$

IV. $\int_{\varkappa}^{y}(f(\alpha))^{d\alpha} = \int_{\varkappa}^{\mu}(f(\alpha))^{d\alpha} \cdot \int_{\mu}^{y}(f(\alpha))^{d\alpha}, \varkappa \leq \mu \leq y$.

V. $\int_{\varkappa}^{\varkappa}(f(\alpha))^{d\alpha} = 1$, $\int_{\varkappa}^{y}(f(\alpha))^{d\alpha} = \left(\int_{y}^{\varkappa}(f(\alpha))^{d\alpha}\right)^{-1}$.

III. The Definition of Exponential Type Multiplicatively (s, P)-Functions

In this section we give a new definition, which is called exponential type multiplicatively (s, P)-function and study some of its basic algebraic properties.

Definition 3. A positive function $f: \mathfrak{I} \to \mathbb{R}$ is called exponential type multiplicatively (s, P)-function if,

$$f(\omega\varkappa + (1-\omega)y) \leq (f(\varkappa)f(y))^{e^{s\omega}+e^{s(1-\omega)}-2}$$

holds for all $\varkappa, y \in \mathfrak{I}$ and $\omega \in [0,1]$ and $s \in [\ln 2.5, 1]$.

Note that for $s = 1$, the class of exponential type multiplicatively (s, P)-functions reduces to the class of exponential type multiplicatively P-functions.

Remark 1. The range of the exponential type multiplicatively (s, P)-functions is $[1, \infty)$.

Using the definition of the exponential type multiplicatively (s, P)-function for $\omega = 1$ and $c \in I$, we have

$$f(c) \leq (f(c))^{e^{s\omega}+e^{s(1-\omega)}-2}$$

$$(f(c))^{e^{s\omega}+e^{s(1-\omega)}-3} \geq 1$$

$$f(c) \geq 1$$

which is the desired result.

Lemma 1. For all $\omega \in [0,1]$ and $s \in [\ln 2.5, 1]$ the following inequality holds.

$$e^{s\omega}+e^{s(1-\omega)}-2\geq 1.$$

Proof. The proof is clear from the inequalities

$$e^{s\omega}-1\geq \omega \text{ and } e^{s(1-\omega)}-1\geq 1-\omega.$$

Proposition 3. Every multiplicatively (s,P)-function is an exponential type multiplicatively (s,P)-function.

Proof. According to *Lemma 1* and *Definition 3*, we have

$$f(\omega\varkappa+(1-\omega)y)\leq (f(\varkappa)f(y))^{\omega^s+(1-\omega)^s}$$
$$\leq (f(\varkappa)f(y))^{e^{s\omega}+e^{s(1-\omega)}-2},$$

which completes the proof.

Theorem 4. Let $f,g:[\varkappa,y]\to\mathbb{R}$. If f and g are exponential type multiplicatively (s,P)-functions, then $f.g$ is exponential type multiplicatively (s,P)-function.

Proof. Let f and g be exponential type multiplicatively (s,P)-functions. Then

$$\begin{aligned}
&(f\cdot g)(\omega\varkappa+(1-\omega)y)\\
&= f(\omega\varkappa+(1-\omega)y)\cdot g(\omega\varkappa+(1-\omega)y)\\
&\leq (f(\varkappa)f(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\cdot (g(\varkappa)g(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&= (f(\varkappa)g(\varkappa))^{e^{s\omega}+e^{s(1-\omega)}-2}(f(y)g(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&= ((f\cdot g)(\varkappa))^{e^{s\omega}+e^{s(1-\omega)}-2}\cdot ((f\cdot g)(y))^{e^{s\omega}+e^{s(1-\omega)}-2}.
\end{aligned}$$

Thus, the proof is completed.

Theorem 5. If $f:\mathfrak{J}\to\mathfrak{S}$ is convex and $g:\mathfrak{S}\to\mathbb{R}$ is an exponential type multiplicatively (s,P)-function and nondecreasing, then $g\circ f:\mathfrak{J}\to\mathbb{R}$ is an exponential type multiplicatively (s,P)-function.

Proof. For $\varkappa,y\in\mathfrak{J}$ and $\omega\in[0,1]$, we get

$$\begin{aligned}
&(g\circ f)(\omega\varkappa+(1-\omega)y)\\
&= gf(\omega\varkappa+(1-\omega)y)\\
&\leq g(\omega f(\varkappa)+(1-\omega)f(y))\\
&\leq [g(f(\varkappa))g(f(y))]^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&= [(g\circ f)(\varkappa)]^{e^{s\omega}+e^{s(1-\omega)}-2}.[(g\circ f)(y)]^{e^{s\omega}+e^{s(1-\omega)}-2}.
\end{aligned}$$

This completes the proof.

Theorem 6. Let $f_i:[\varkappa,y]\to\mathbb{R}$ be an arbitrary family of exponential type multiplicatively (s,P)-functions and let $f(\alpha)=\sup_i f_i(\alpha)$. If $I=\{\theta\in[\varkappa,y]: f(\theta)<\infty\}\neq\emptyset$, then I is an interval and f is an exponential type multiplicatively (s,P)-function on I.

Proof. For all $\varkappa,y\in I$ and $\omega\in[0,1]$, we have

$$\begin{aligned}
&f(\omega\varkappa+(1-\omega)y)\\
&= \sup_i f_i(\omega\varkappa+(1-\omega)y)\\
&\leq \sup_i\left[(f_i(\varkappa)f_i(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\right]\\
&\leq \sup_i(f_i(\varkappa))^{e^{s\omega}+e^{s(1-\omega)}-2}\cdot\sup_i(f_i(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&= (f(\varkappa))^{e^{s\omega}+e^{s(1-\omega)}-2}\cdot (f(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&< \infty.
\end{aligned}$$

So, the proof is completed.

Theorem 7. If $f:[\varkappa,y]\to\mathbb{R}$ is exponential type multiplicatively (s,P)-function, then f is bounded on $[\varkappa,y]$.

Proof. Let $\alpha\in[\varkappa,y]$ and $K=\max\{f(\varkappa),f(y)\}$. Then, there exists $\omega\in[0,1]$ such that $\alpha=\omega\varkappa+(1-\omega)y$. Thus, since $e^{s\omega}\leq e$ and $e^{s(1-\omega)}\leq e$, we have

$$\begin{aligned}
f(\alpha)&\leq f(\omega\varkappa+(1-\omega)y)\\
&\leq (f(\varkappa)f(y))^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&\leq K^{e^{s\omega}+e^{s(1-\omega)}-2}\\
&\leq K^{2(e-1)}\\
&= M
\end{aligned}$$

Also, for all $\alpha\in[\varkappa,y]$, there exists $\mu\in\left[0,\frac{y-\varkappa}{2}\right]$ such that $\alpha=\frac{\varkappa+y}{2}+\mu$ or $\alpha=\frac{\varkappa+y}{2}-\mu$. Without loss of generality, we suppose $\alpha=\frac{\varkappa+y}{2}+\mu$. So, we have

$$f\left(\frac{\varkappa+y}{2}\right)=f\left(\frac{1}{2}\left[\frac{\varkappa+y}{2}+\mu\right]+\frac{1}{2}\left[\frac{\varkappa+y}{2}-\mu\right]\right)$$
$$\leq\left[f(\alpha)\cdot f\left(\frac{\varkappa+y}{2}-\mu\right)\right]^{\sqrt{e^s}-1}.$$

Since M is the upper bound, we have

$$f(\alpha) \geq \frac{\left[f\left(\frac{x+y}{2}\right)\right]^{\frac{1}{\sqrt{e^s-1}}}}{f\left(\frac{\varkappa+y}{2}-\mu\right)}$$

$$\geq \frac{\left[f\left(\frac{x+y}{2}\right)\right]^{\frac{1}{\sqrt{e^s-1}}}}{M}$$

$$= m.$$

IV. HERMITE-HADAMARD'S INEQUALITY FOR EXPONENTIAL TYPE MULTIPLICATIVELY (s, P)-FUNCTIONS

Theorem 8. Let f be an exponential type multiplicatively (s, P)-function on $[\varkappa, y]$. Then

$$f\left(\frac{\varkappa+y}{2}\right) \leq \left(\int_{\varkappa}^{y} f(\alpha)^{d\alpha}\right)^{\frac{2(e^s-2)}{y-\varkappa}}$$

$$\leq [f(\varkappa)f(y)]^{\frac{4(e^s-2)(e^s-s-1)}{s}} \quad (1)$$

The above inequality is called Hermite-Hadamard integral inequality for exponential type multiplicatively (s, P)-functions.

Proof. Note that

$$\ln f\left(\frac{\varkappa+y}{2}\right)$$

$$= \ln\left(f\left(\frac{\omega\varkappa+(1-\omega)y+\omega y+(1-\omega)\varkappa}{2}\right)\right)$$

$$= \ln\left(f\left(\frac{\omega\varkappa+(1-\omega)y}{2}+\frac{\omega y+(1-\omega)\varkappa}{2}\right)\right)$$

$$\leq \ln\left[\left(f\left(\frac{\omega\varkappa+(1-\omega)y}{2}\right)\right)\left(f\left(\frac{\omega y+(1-\omega)\varkappa}{2}\right)\right)^{e^s-2}\right]$$

$$= (e^s-2)\ln f\left(\frac{\omega\varkappa+(1-\omega)y}{2}\right) + (e^s-2)\ln f\left(\frac{\omega y+(1-\omega)\varkappa}{2}\right).$$

Integrating the above inequality with respect to ω on [0,1], we have

$$\ln f\left(\frac{\varkappa+y}{2}\right)$$

$$\leq (e^s-2)\int_0^1 \ln f\left(\frac{\omega\varkappa+(1-\omega)y}{2}\right)d\omega$$

$$+(e^s-2)\int_0^1 \ln f\left(\frac{\omega y+(1-\omega)\varkappa}{2}\right)d\omega$$

$$= (e^s-2)\left[\frac{1}{y-\varkappa}\int_{\varkappa}^{y}\ln f(\alpha)d\alpha + \frac{1}{\varkappa-y}\int_{y}^{\varkappa}\ln f(\alpha)d\alpha\right]$$

$$= \frac{2(e^s-2)}{y-\varkappa}\int_{\varkappa}^{y}\ln f(\alpha)d\alpha$$

Thus, we have

$$f\left(\frac{\varkappa+y}{2}\right) \leq e^{\left(\frac{2(e^s-2)}{y-\varkappa}\int_{\varkappa}^{y}\ln f(\alpha)d\alpha\right)}$$

$$= \left(\int_{\varkappa}^{y} (f(\alpha))^{d\alpha}\right)^{\frac{2(e^s-2)}{y-\varkappa}}.$$

Hence,

$$f\left(\frac{\varkappa+y}{2}\right) \leq \left(\int_{\varkappa}^{y} (f(\alpha))^{d\alpha}\right)^{\frac{2(e^s-2)}{y-\varkappa}}$$

To prove the right hand side of the inequality (1), we have

$$\left(\int_{\varkappa}^{y} (f(\alpha))^{d\alpha}\right)^{\frac{1}{y-\varkappa}}$$

$$= \left(e^{\left(\int_{\varkappa}^{y}\ln f(\alpha)d\alpha\right)}\right)^{\frac{1}{y-\varkappa}}$$

$$= e^{\frac{1}{y-\varkappa}\left(\int_{\varkappa}^{y}\ln f(\alpha)d\alpha\right)}$$

$$= e^{\left(\int_0^1 \ln f(\omega\varkappa+(1-\omega)y)d\omega\right)}$$

$$\leq e^{\int_0^1 \ln (f(\varkappa)f(y))^{e^{s\omega}+e^{s(1-\omega)}-2}d\omega}$$

$$= e^{\int_0^1 [(e^{s\omega}+e^{s(1-\omega)}-2)\ln[f(\varkappa)f(y)]]d\omega}$$

$$= e^{\ln[f(\varkappa)f(y)]^{\frac{2(e^s-s-1)}{s}}}$$

$$= [f(\varkappa)f(y)]^{\frac{2(e^s-s-1)}{s}}.$$

Hence,

$$\left(\int_{\varkappa}^{y} (f(\alpha))^{d\alpha}\right)^{\frac{1}{y-\varkappa}} \leq [f(\varkappa)f(y)]^{\frac{2(e^s-s-1)}{s}}.$$

So, the proof is completed.

Remark 2. For $s = 1$, the inequality (1) reduces to the following inequality for exponential type multiplicatively P-functions.

$$f\left(\frac{\varkappa+y}{2}\right)\leq\left(\int_{\varkappa}^{y}f(\alpha)^{d\alpha}\right)^{\frac{2(\mathrm{e}-2)}{y-\varkappa}}$$

$$\leq[f(\varkappa)f(y)]^{4(\mathrm{e}-2)^2}.$$

Corollary 1. Let f and g be two exponential type multiplicatively (s,P)-functions on $[\varkappa,y]$. Then

$$f\left(\frac{\varkappa+y}{2}\right)g\left(\frac{\varkappa+y}{2}\right)$$

$$\leq\left(\int_{\varkappa}^{y}(f(\alpha))^{d\alpha}\int_{\varkappa}^{y}(g(\alpha))^{d\alpha}\right)^{\frac{2(\mathrm{e}-2)}{y-\varkappa}}$$

$$\leq[f(\varkappa)f(y)g(\varkappa)g(y)]^{\frac{4(\mathrm{e}-2)(e^s-s-1)}{s}}.$$

Corollary 2. Let f and g be two exponential type multiplicatively (s,P)- functions on $[\varkappa,y]$. Then

$$\frac{f\left(\frac{\varkappa+y}{2}\right)}{g\left(\frac{\varkappa+y}{2}\right)}\leq\left(\frac{\int_{\varkappa}^{y}(f(\alpha))^{d\alpha}}{\int_{\varkappa}^{y}(g(\alpha))^{d\alpha}}\right)^{\frac{2(\mathrm{e}-2)}{y-\varkappa}}$$

$$\leq\left(\frac{f(\varkappa)f(y)}{g(\varkappa)g(y)}\right)^{\frac{4(\mathrm{e}-2)(e^s-s-1)}{s}}.$$

V. Conclusion

In this paper, we investigated exponential type multiplicatively multiplicative (s,P)-functions. We obtained a new version of the Hermite–Hadamard type integral inequality in the setting of multiplicative calculus for the newly classes of functions. We also derived integral inequalities of Hermite–Hadamard type for the multiplication and division of exponential type multiplicatively multiplicative (s,P)-functions in multiplicative calculus. In recent years, Hermite–Hadamard inequalities have grown into a significant tool for mathematical analysis, probability theory, optimization, and other fields of mathematics. Many studies have been dedicated to bringing a new dimension to the theory of inequalities. We believe that this class of functions will be deeply researched in this attractive and absorbing field of inequalities and also in different areas of pure and applied sciences. We also believe that our techniques and ideas will stimulate further research in this field.

References

[1] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite–Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[2] J. E. Pecaric, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[3] M. A. Ali, M. Abbas, Z. Zhang, I. B. Sial, and R. Arif, “On integral inequalities for product and quotient of two multiplicatively convex functions”, *Asian Res. J. Math.*, vol. 12, no. 3, pp. 1–11, 2019.

[4] M. Kadakal, H. Karaca and İ. İşcan, “Hermite-Hadamard type inequalities for multiplicatively geometrically P-functions”, *Poincare J. Anal. Appl.* Vol. 2, pp. 77–85, 2018.

[5] M. A. Latif and T. Du, “Hermite-Hadamard type inequalities for harmonically-convex functions using fuzzy integrals”, *Filomat*, vol.36, no.12, pp. 4099–4110, 2022.

[6] S. Maden, S. Turhan and İ. İşcan, “Hermite-Hadamard inequality for strongly p-convex function”, *Turkish J. Math. Comp. Sci.*, vol.10, pp. 184–189, 2018.

[7] M. A. Noor, F. Qi and M. U. Awan, “Some Hermite-Hadamard type inequalities for log−h−convex functions”, *Analysis*, vol. 33, pp. 1–9, 2013.

[8] S. Özcan, “Hermite-Hadamard type inequalities for exponential type multiplicatively convex functions”, *Filomat*, vol. 37, no.28, pp. 9777–9789, 2023.

[9] S. Özcan, “Hermite-Hadamard type inequalities for m-convex and (α,m)-convex functions”, *J. Inequal. Appl.*, 175 (2020), 2020.

[10] S. Özcan, “Hermite-Hadamard type inequalities for multiplicative h-convex functions”, *Konuralp J. Math*., vol. 8, no.1, pp. 158–164, 2020.

[11] S. Özcan, “Hermite–Hadamard type inequalities for multiplicative h-preinvex functions”, *Turk. J. Anal. Number Theory*, vol. 9, no.3, pp. 65–70, 2020.

[12] S. Özcan, “ On refinements of some integral inequalities for differentiable prequasiinvex functions”, *Filomat*, vol. 33, no.14, pp. 4377–4385, 2019.

[13] M. E. Özdemir, H. K. Önalan and M. A. Ardıç, “Hermite-Hadamard type inequalities for (h(α,m))-convex functions”, *J. Conc. Appl. Math*., vol. 13, no.1, pp. 96–107, 2015.

[14] S. Numan and İ. İşcan, “On exponential type P-functions”, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat*,. vol. 70, no. 1, pp. 497–509, 2021.

[15] A. E. Bashirov, E. Kurpınar and A. Özyapıcı, “Multiplicative calculus and applications”, *J. Math. Anal. Appl.*, vol. 337, no.1, pp. 36–48, 2008.